Inductance and Oscillations

Everything is moving ahead on schedule, right over the cliff.
— Henry Kissinger

Magnetic energy and inductance

In the same way that a capacitor stores energy in the E-field, a device that creates a B-field when a current passes through it can store energy in the B-field. Such a device is usually called an inductor.

A solenoid provides a simple example. We consider a solenoid of cross-section area A and n turns per unit length carrying current I. We are interested in the work done by external agents as the current builds up; this represents the energy put into (and stored in) the magnetic field.

At any instant, the magnetic flux through one turn of the solenoid is $\Phi_m = BA$. By the flux rule the emf induced in this turn is $\mathcal{E}_1 = -A \cdot dB/dt$. A length d of the solenoid contains nd turns in series, which act like batteries in series, so their emf's add, giving a total emf induced in this length $\mathcal{E} = -nAd \cdot dB/dt$. The B-field has magnitude $B = \mu_0 nI$, so it follows that the total emf in length d of the solenoid is

$$\mathcal{E} = -\mu_0 n^2 Ad \frac{dI}{dt} .$$

This formula shows that the emf is (minus) a geometric constant times the rate of change of the current. The geometric constant is called the **self-inductance**:

Self inductance	The induced emf in an element due to a changing current through the element is given by $\mathcal{E} = -L\dfrac{dI}{dt}$, where L is the self inductance of the element.

For the solenoid, we have thus found that the self-inductance of length d is $L = \mu_0 n^2 Ad$.

The negative sign (Lenz's law) shows that if the current is increasing the potential *drops* as we follow the current through the inductor. This means that power is being delivered *to* the inductor, in an amount equal to I times the potential drop, or

$$P = I \cdot L \frac{dI}{dt} = \frac{d}{dt}\left(\tfrac{1}{2}LI^2\right).$$

Integrating with respect to time, we find for the energy delivered to the inductor as the current builds from zero to its final value I:

Energy in an inductor	$U_m = \tfrac{1}{2}LI^2$

This energy is stored in the magnetic field. For the section of length d of the solenoid we have been considering, the amount of energy stored is

$$U_m = \tfrac{1}{2}\mu_0 n^2 I^2 \cdot Ad = \frac{B^2}{2\mu_0} \cdot Ad$$

Since Ad is the volume of this section of the solenoid, we can interpret this as the product of a volume and an energy per unit volume, given by:

Magnetic energy density	$u_m = \dfrac{B^2}{2\mu_0}$

As in the electric case, the energy density is proportional to the square of the field magnitude.

The energy density in a strong laboratory magnetic field ($B \sim 1$ T) is about 10^5 times stronger than in a strong lab E-field ($E \sim 10^6$ V/ms). On the other hand, as we will show later, in an electromagnetic wave the two densities are equal.

The large energy density in the B-field is why care must be taken before interrupting the current to a device that operates with a large B-field.

Self-inductance measures the emf induced in an element due to a change in the current through the *same* element. (It is often called the "back" emf, because by Lenz's law it opposes the change in the current.) But if two different elements are close enough together that the magnetic field of one influences the other, then a change in current in the first can induce an emf in the second. The strength of this effect is measured by the **mutual inductance**, a geometric constant M_{12} defined by:

$$\mathcal{E}_{1/2} = -M_{12}\frac{dI_2}{dt}.$$

Here $\mathcal{E}_{1/2}$ is the emf induced in element 1 by a change in current in element 2, and $M_{1/2}$ is the mutual inductance between these elements.

The phenomenon of mutual inductance is the basis of devices such as transformers, which pass electromagnetic energy from one circuit to another.

L-R circuits

Establishment of a B-field by a current must take some time, since energy is never transferred instantly. We will analyze this situation using a simple circuit consisting of a battery, a resistor and an inductor.

Shown is the circuit. The switch is closed at $t = 0$. At the instant when the current is I, the loop rule gives

$$\mathcal{E} - L\frac{dI}{dt} = IR.$$
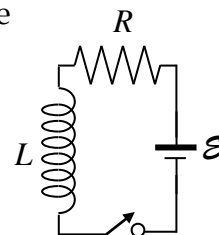
Rearranging, we find

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{\mathcal{E}}{L}.$$

This is much like the differential equation for the R-C circuit, so the solution is similar:

$$I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-(R/L)t}\right)$$

The current rises to approach its maximum value $\mathcal{E}/R$, at a rate controlled by the "time-constant" L/R.

If, after a current is established, the switch is again opened, the current dies away and the B-field collapses. The rate of this exponential decay is again controlled by L/R.

The L-C-R oscillator

A capacitor stores energy in its E-field, and an inductor stores energy in its B-field. If these elements are connected in series in a circuit, they can pass the energy back and forth between them. This periodic exchange of energy constitutes an electromagnetic oscillator. Such oscillators are important parts of many electronic circuits.

In the circuit shown we start with the capacitor charged to an initial charge Q_0 before the switch is closed.

When the switch is closed (at $t = 0$) the capacitor starts to discharge through the rest of the circuit. The loop rule gives

$$\frac{Q}{C} - L\frac{dI}{dt} = IR.$$
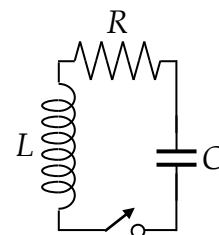
Here $I = -dQ/dt$, since the capacitor is discharging. Substituting, we find

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = 0.$$

This equation has the same mathematical form as the equation for a mechanical oscillator with damping. If the resistance (the "damping" that dissipates energy) is not too large, the charge oscillates with time, but with decreasing magnitude. A typical case of Q vs. t is shown.
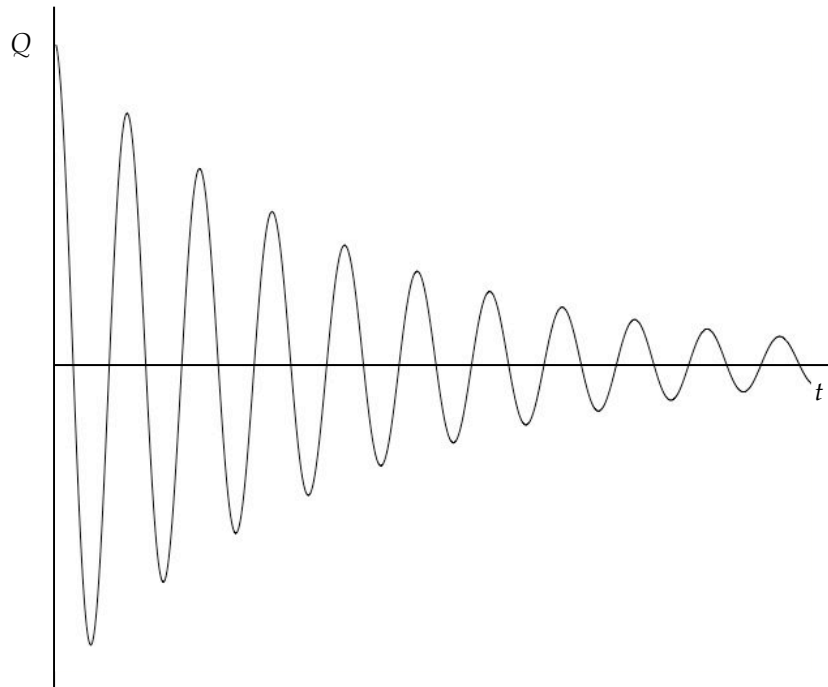
The frequency of the oscillation is determined by the values of L, C and R. For the "free" case where $R = 0$ there is a simple formula:

$$\omega_0 = \frac{1}{\sqrt{LC}} \ .$$

The energy is entirely in the capacitor twice per cycle and entirely in the inductor twice per cycle.

To keep the oscillation going when there is resistance, energy must be put into the circuit. We discuss this later in connection with AC circuits.